The Mathematics Behind Einstein's Theory of Relativity

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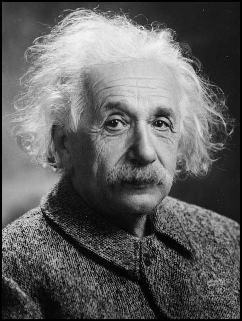
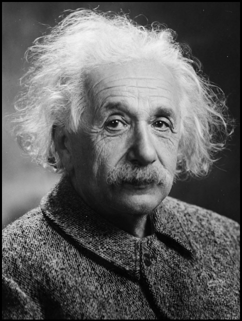
The Wonderful World of Maths, Taster Day
4 April, 2017

Who Is He?



Who Is He?



Einstein in 1947

Albert Einstein, physicist, 1879-1955

- 1905: Discovered special relativity.
- 1915: Discovered general relativity.

Awarded Nobel prize in 1921.

- (1905: Discovery of the photoelectric effect.)

Photo by O. J. Turner. From the U.S. Library of Congress

Why Am I Here?

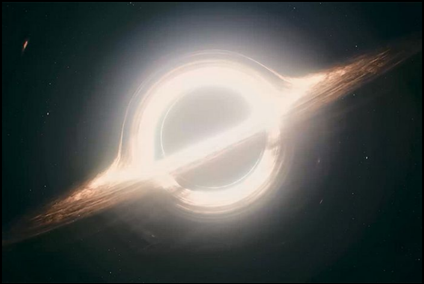


Image of black hole from the movie *Interstellar* (Paramount).

Theory of relativity:

- Revolutionised modern physics.
- Involved advanced maths.

Goal: Introduce maths behind:

1. Special relativity
2. General relativity

Einstein's Postulates

(A. Einstein, 1905) Postulates of special relativity:*

1. *The laws of physics are the same in all inertial frames of reference.*
2. *The speed of light in vacuum has the same value in all inertial frames of reference.*

Postulates ⇒ strange physical consequences

Two observers moving at different velocities will:

1. Perceive different events to be "at the same time".
2. Measure different lengths for the same object.

Q. What is the mathematical explanation?

Image from `clipartall.com`

* Quoted from Nobelprize.org.

Minkowski's Contribution

(Hermann Minkowski, 1907):

- Mathematical formulation of special relativity.
- In terms of geometry.

Minkowski died soon after, in 1909.

- But his geometric viewpoint eventually led to ...
- ... Einstein's theory of general relativity.

H. Minkowski (1864–1909)

Photo from www.spacetimesociety.org.

Spacetime

Classical physics—separate notions of:

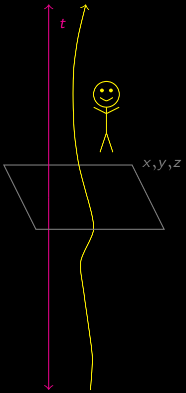
- (3-d) space $\mathbb{R}^3$: contains points $p = (x, y, z)$.
- (1-d) time $\mathbb{R}$: contains real numbers t.

Relativity—combined notion of spacetime.

- *Notions of space and time cannot be separated.*
- (4-d) spacetime $\mathbb{R}^4$: contains events $P = (t, x, y, z)$.

Event P: "a point in space at a given time".

- Observer: *curve* in spacetime (worldline).



My worldline (yellow) in spacetime.

Classical Length and Distance

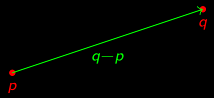
Consider two points in space:

$$p = (p_x, p_y, p_z), \qquad q = (q_x, q_y, q_z).$$

$\vec{pq} = q - p$: vector from p to q.

- $|q-p|$: distance from p to q.
- Squared distance from p to q:

$$|q-p|^2 = (q_x - p_x)^2 + (q_y - p_y)^2 + (q_x - p_x)^2.$$



(Green) 3-vector from p to q.

Relativistic "Length" and "Distance"

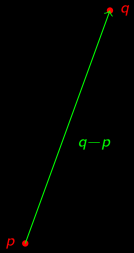
Consider two events in spacetime:

$$p = (p_t, p_x, p_y, p_z), \qquad q = (q_t, q_x, q_y, q_z).$$

Now define "squared distance" from p to q by

$$\|q - p\|_m^2 = -(q_t - p_t)^2 + (q_x - p_x)^2 + (q_y - p_y)^2 + (q_x - p_x)^2.$$

- Similar to previous squared distance...
- ... but *flip the sign in the t-component*!



(Green) 4-vector from p to q.

And... the Dot Product

Dot product in space: for 3-vectors $\vec{u} = (u_x, u_y, u_z)$ and $\vec{v} = (v_x, v_y, v_z)$:

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z .$$

- Captures lengths and angles.
- Generates (the usual) geometry on 3-dimensional Euclidean space.

Spacetime product: for 4-vectors $\vec{u} = (u_t, u_x, u_y, u_z)$, $\vec{v} = (v_t, v_x, v_y, v_z)$:

$$m(\vec{u}, \vec{v}) = -u_t v_t + u_x v_x + u_y v_y + u_z v_z .$$

- $m(\vec{u}, \vec{u})$ is the "weird squared length" of $\vec{u}$.

Minkowski Geometry

Weird product $\Rightarrow$ weird geometry on spacetime $\mathbb{R}^4$

- Called Minkowski geometry.
- Formally, described by the Minkowski metric:
$$m = -dt^2 + dx^2 + dy^2 + dz^2 .$$
- (Here, we assumed units with speed of light $c = 1$.)

Mathematically, we take this geometric viewpoint:

- Special relativity $\Leftrightarrow$ Minkowski geometry.

Causal Character I

$m(\vec{u}, \vec{u})$ can now be positive, negative, or zero!

- Each case has its own interpretation.

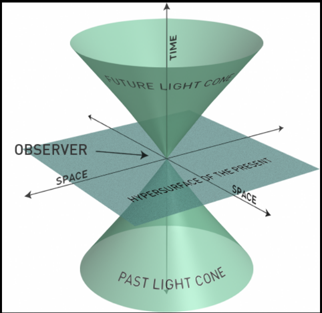


Image by Stib. From en.wikipedia.org.

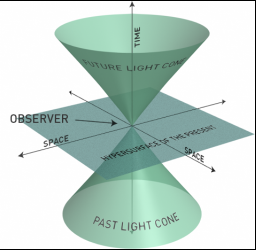
(1) $\vec{u}$ is spacelike: $m(\vec{u}, \vec{u}) > 0$

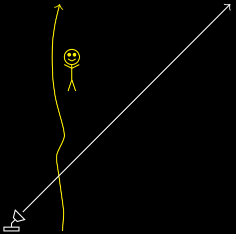
- Points from origin to outside light cone.
- Represents spatial direction.
- Measures (squared) length/distance.

(2) $\vec{u}$ is null: $m(\vec{u}, \vec{u}) = 0$

- Lies on light cone.
- Represents light ray.

Causal Character II





(3) $\vec{u}$ is timelike: $m(\vec{u}, \vec{u}) < 0$

- Points from origin to inside light cone.
- Directions for observer worldlines.
- Measures (squared) time elapsed.

Light rays are null lines.

- Bottom image: white ray.

Observers are timelike curves.

- Bottom image: yellow curve.
- *An observer can never travel faster than the speed of light!* :(

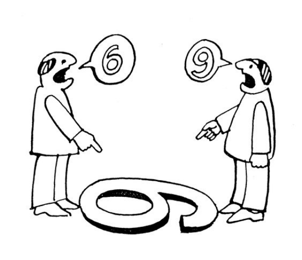
Relativity

Minkowski geometry is weird.

- Leads to interesting physical consequences.

Many things cannot be measured absolutely:

- Examples: elapsed time, length, energy-momentum.
- Only makes sense relative to an observer.



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Inertial Frames

Consider inertial observers A, B.

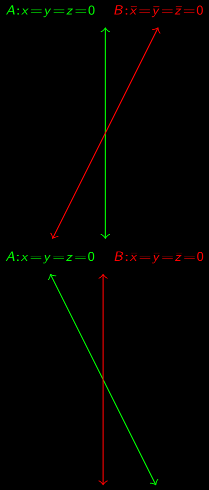
- Moving with constant velocities.

Frame of reference (t, x, y, z) about A:

- $x = y = z = 0$ along A.
- A stationary.
- B moving relative to A.

Frame of reference $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ about B:

- $\bar{x} = \bar{y} = \bar{z} = 0$ along B.
- B stationary.
- A moving relative to B.

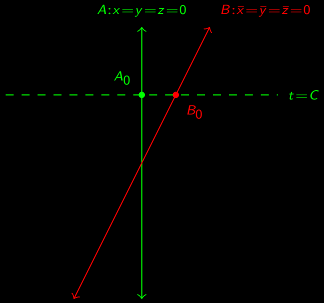


Simultaneity I

Observers moving at different velocities see different events as "at the same time."

Consider event A_0 on A's worldline.

- What A sees as simultaneous to A_0 is $t = C$...
- ...i.e., space (m-)perpendicular to A at A_0.
- To A, events A_0 and B_0 are simultaneous.



Simultaneity II

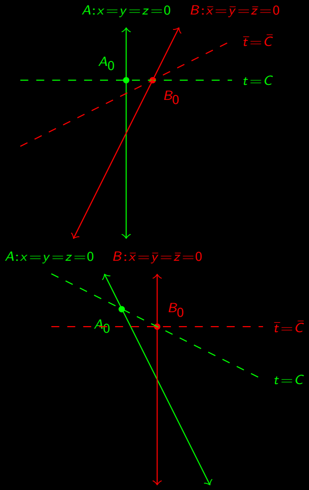
But B sees differently!

- What B sees as simultaneous to B_0 is $\bar{t} = \bar{C}$...
- ...i.e., space (m-)perpendicular to B at B_0.

But remember: this geometry is weird!

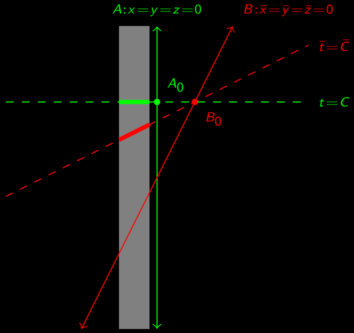
- m-product is quite different.
- $\bar{t} = \bar{C}$ does not hit A_0.

To B, events A_0 and B_0 not simultaneous!



Length Contraction I

Observers moving at different velocities perceive lengths differently.



Shaded region represents rod.

- A stationary with respect to rod.
- B moving with respect to rod.

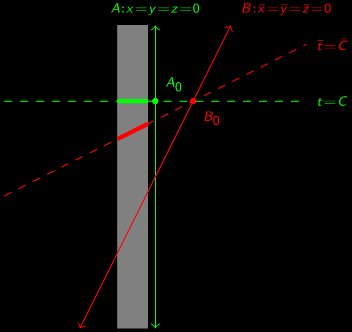
Both A and B measure length of rod.

- A: length of green bolded segment.
- B: length of red bolded segment.

Length Contraction II

Note: A and B measure different lengths!

- **Q.** Who measures the longer length?
- (Hint: It's a trick question.)



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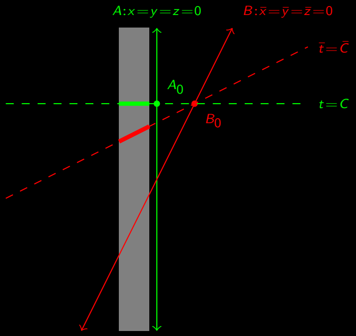


B's measurement consists of:

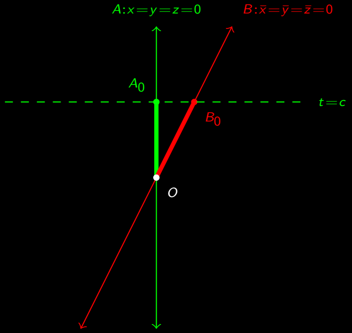
- A's measurement...
- ... + timelike component, ...
- ... which has opposite sign!

B measures shorter length than A.

Image from `clipartkid.com`

Time Dilation

Clocks moving at different velocities observed to tick at different speeds.



Both A and B carry a clock.

- Clocks synchronised at O.
- A measures both clocks at $t = C$.
- **Q.** What will A see?

A measures less time elapsed on B's clock than A's clock.

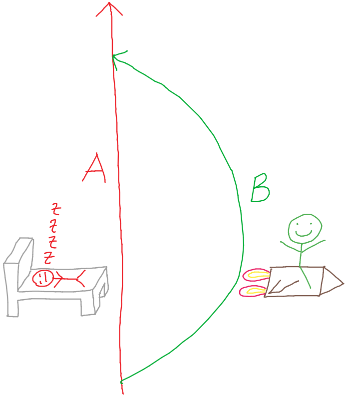
Twin Paradox I

Twin paradox: classic thought experiment from special relativity.

Consider twins, A and B.

- A goes to sleep for a long time.
- B flies off in a rocketship.
- B eventually flies back to A.

Q. Who ages more? A or B?



Twin Paradox II

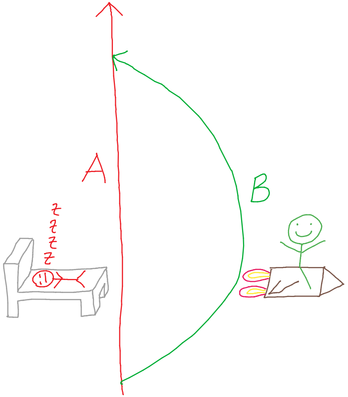
Times elapsed for A and B:

- A: (m-)length of red segment.
- B: (m-)length of green segment.

B-segment is shorter than A-segment.

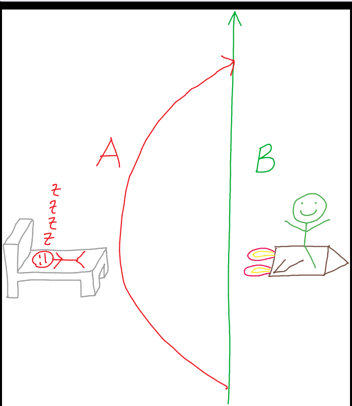
- *When B returns to A...*
- *... A will have aged more than B.*

Two different curves joining two events will have different lengths.



Twin Paradox III

OK, so why was this a paradox?

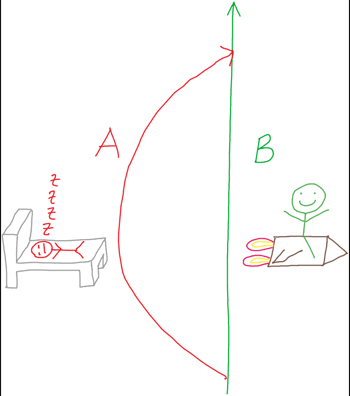


Consider *B*'s frame of reference:

-  *By same reasoning as before, shouldn't B have aged more than A?!*

Twin Paradox III

OK, so why was this a paradox?



Consider B's frame of reference:

- *By same reasoning as before, shouldn't B have aged more than A?!*

Only the first argument is correct:

- Lengths of A's and B's curves are independent of frames of reference.
- B's frame of reference is not inertial, so the spacetime geometry looks quite different from B's point of view.

Classical Gravity

Problem: special relativity does not include gravity.

Newtonian gravity: attractive force between two particles.

- Not compatible with special relativity.

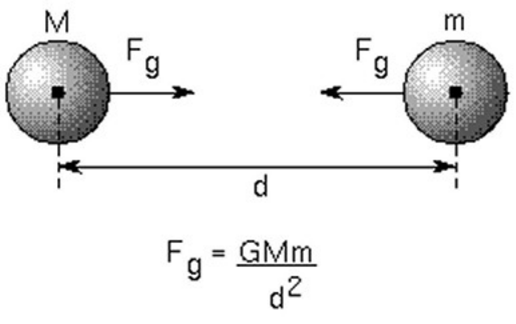


Image from Astronomy Notes
astronomynotes.com/gravappl/s3.htm

Geometry and Gravity

(Einstein, 1915) General relativity

Revolutionary view of gravity.

- *Not modeled as a force...*
- *... but as curvature of spacetime.*

Simplified viewpoint (image):

- Object introduces gravity...
- ... by bending the spacetime itself.

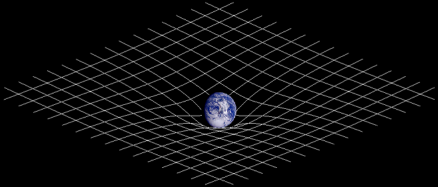


Image by Johnstone on en.wikipedia.org.

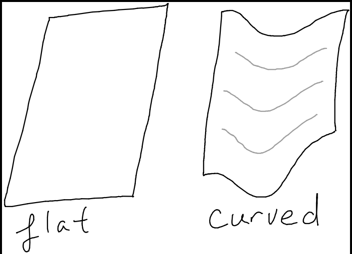
Curved Spacetimes

Setting of special relativity:

- Minkowski spacetime $\mathbb{R}^4$.
- Has strange geometry, but still flat.

Setting of general relativity:

- Curved spacetimes $\mathcal{M}$.
- Gravity manifested in the geometry of $\mathcal{M}$.



Spacetime: formally modeled as Lorentzian manifold.

- Curved 4-dimensional object.
- 1 timelike (negative) direction, 3 spacelike (positive) directions.

Geodesics and Light

Geodesics: analogues in curved spacetimes of lines.

- Light rays modeled by null geodesics.

General relativity predicts light should bend.

- Confirmed by Eddington in 1919.
- Studied positions of stars passing near the sun during solar eclipse.

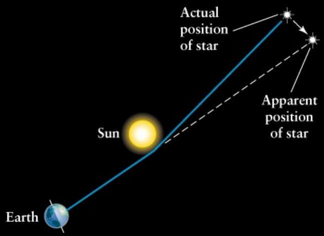


Image from `frigg.physastro.mnsu.edu/~eskridge/astr101/kauf24_5.JPG`.

The Einstein Field Equations

Q. How are gravity and matter related?

The Einstein Field Equations

Q. How are gravity and matter related?

A. They are coupled together via the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}.$$

Left-hand side: gravitational content

- Related to curvature of spacetime.
- Λ: cosmological constant

Right-hand side: matter content

- T: stress-energy tensor associated with matter fields.

Solving the Einstein Equations

Question

Q. *What do we do with the Einstein field equations?*

Solving the Einstein Equations

Question

Q. *What do we do with the Einstein field equations?*

The Einstein equations can be viewed as partial differential equations.

- Equations containing unknown functions and their derivatives.

Given initial conditions, we can (in theory) solve the Einstein field equations for the spacetime (i.e., universe) itself!

- Roughly, *if we know the state of the universe "at a given time", then we can (in theory) predict the past and future!*
- Of course, in practice, doing this is really, really hard. :(

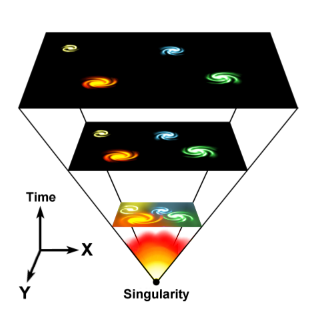
The Big Bang Singularity

To simplify, assume spacetime is:

- Homogeneous ("looks the same everywhere")
- Isotropic ("same in all directions")

Solve Einstein equations *backwards*.

- (Coupled to "dust" matter.)
- $\Rightarrow$ Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes (1920s, 1930s).



After *finite* elapsed time, universe "shrinks down to a point".

- Early model of big bang singularity.

Image from ScienceBlogs. (`scienceblogs.com/startswithabang/2010/04/05/did-the-universe-start-from-a/`)

Other Bad Things

Solving the Einstein equations forward:

- Sometimes, gravity (curvature) can become extremely strong in a region.

The spacetime can have a black hole:

- Once light passes a boundary into this region (event horizon)...
- ... it can no longer escape this region.

Other Bad Things

Solving the Einstein equations forward:

- Sometimes, gravity (curvature) can become extremely strong in a region.

The spacetime can have a black hole:

- Once light passes a boundary into this region (event horizon)...
- ... it can no longer escape this region.

The spacetime can also collapse with a singularity:

- Spacetime geometry collapses prematurely.
- An observer can, after finite elapsed time, ...
- ... reach the singularity and no longer exist.

An Application

Q. What *practical* things come from relativity?

An Application

Q. What *practical* things come from relativity?

One example is GPS (global positioning system).

GPS used to determine your location:

- Receives signals from multiple satellites.
- Compares time difference between signals.

For GPS to be precise enough to be useful (within ~ 10 metres):

- Need to account for relativistic effects.
- Special relativity: orbiting satellites moving with respect to earth.
- General relativity: satellites experience less gravity than on earth.

Thank You!

Thank you for your attention!