MTH6126 **Metric Spaces**

Assignment 1 **Due on Thursday 3rd October at 12 noon**

Write your name and student number at the top of your assignment before handing it in. Staple all pages together. Post your answers to **question 4** *and* **question 5** *in the Orange Post-Box on the second floor of the Maths building.*

1. For each of the following subsets of $\mathbb{R}$, determine the supremum, if it exists, and decide whether it is a maximum. Justify your answers.

   (a) $\{x : x^2 - 2 < 0\}$,

   (b) $\{x^2 - 2 : -2 \leq x < 2\}$,

   (c) $\{1 - 1/n^2 : n = 1, 2, 3, \ldots\}$ and

   (d) $\{1 + 1/n^3 : n = 1, 2, 3, \ldots\}$.

   Is it **silly** to pose the following problems?

   (a) "Determine the maximum of a set S of real numbers, and decide whether or not it is a supremum" .

   (b) "Determine the maximum and the supremum of a set S of real numbers" .

   (c) "Determine the supremum of a set S of real numbers, and decide whether or not it is a maximum" .

   Give reasons for your answer to each of (a), (b) and (c).

2. Suppose (x_n) and (y_n) are Cauchy sequences in $\mathbb{R}$. Prove that (z_n), where (i) $z_n = (x_n - y_n)$; (ii) $z_n = x_n . y_n$ for all n, is a Cauchy sequence too.

3. For $x = (0, 0, 2, 3)$ and $y = (-4, 0, -2, 5)$ in $\mathbb{R}^4$, find $d_0(x, y)$, $d_1(x, y)$, $d_2(x, y)$ and $d_\infty(x, y)$.

****

Only question 4 and question 5 will receive comments.

4. Consider the sup-metric on $B[0, 2]$:

$$d(f, g) = \sup_{x \in [0,2]} |f(x) - g(x)|.$$

   Let $f(x) = x^2 + 1$ and $g(x) = 2x$. Evaluate $d(f, g)$.

5. Recall that the Euclidean norm of $p = (p_1, p_2) \in \mathbb{R}^2$ is $\|p\|_2 = \sqrt{p_1^2 + p_2^2}$. For $p, q \in \mathbb{R}^2$ define $d(p, q)$ by

$$d(p,q) = \begin{cases} 0, & \text{if } p = q; \\ \|p\|_2 + \|q\|_2, & \text{otherwise.} \end{cases}$$

Prove that d is a metric on $\mathbb{R}^2$. What is the set $\{q = (q_1, q_2) \in \mathbb{R}^2 | d(\mathbf{0}, q) < 1\}$?

****

6. Let $C(a, b)$ be the set of continuous real-valued functions $f : (a, b) \to \mathbb{R}$ on the open interval (a, b). Criticise the proposal to make $C(a, b)$ into a metric space by equipping it with the "metric" $d(f, g) = \sup_{x \in (a,b)} |f(x) - g(x)|$ for all $f, g : (a, b) \to \mathbb{R}$.