Solutions to 2011-2012 MTH6126 Metric Spaces Final Exam

bookwork 1. a) M1) $d(x,x)=0 \ \forall x \in X$ (2 marks)

$d(x,y)>0 \ \forall x \neq y$

$d(x,y)=d(y,x) \ \forall x,y \in X$ (1 mark)

$d(x,z) \leq d(x,y)+d(y,z) \ \forall x,y,z \in X$ (1 mark)

book work b) A set $U \subseteq X$ is open if $\forall x \in U$ $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. (2 marks)

$B_\epsilon(x) = \{y \in X : d(y,x) < \epsilon\}$

A set $U \subseteq X$ is closed if $X \setminus U$ is open. (1 mark)

done in lecture c) Let $x \in (a,b)$. ~~Then~~

Let $\epsilon = \min(b-x, x-a)$ (1)

Then $B_\epsilon(x) \subseteq (a,b)$. (1)

This implies that (a,b) is open. (1)

2. a) $f: X \to Y$ is continuous if $\forall x \in X$ and $\epsilon > 0$ $\exists \delta_x > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta_x$.

book work

(3)

b) $d_1((x,y,z),(x_0,y_0,z_0))$

$= |x - x_0| + |y - y_0| + |z - z_0|$

(2)

if $d_1((x,y,z),(x_0,y_0,z_0)) < \delta$,

then $|x-x_0|<\delta$, $|y-y_0|<\delta$, $|z-z_0|<\delta$.

similar to examples and exercises

$d_1(f(x,y,z), f(x_0,y_0,z_0))$

$= |2x + 3y + 4z - (2x_0 + 3y_0 + 4z_0)|$

$= |2(x-x_0) + 3(y-y_0) + 4(z-z_0)|$

$\leq 2|x-x_0| + 3|y-y_0| + 4|z-z_0|$.

(2)

Choose $\delta = \frac{\epsilon}{9}$, then

$d_1((x,y,z),(x_0,y_0,z_0)) < \delta$

$\Rightarrow d_1(f(x,y,z), f(x_0,y_0,z_0)) < 2\delta + 3\delta + 4\delta$

$= 9\delta < \epsilon$.

(2)

This implies that f is continuous.

(1)

3. a) $f_n \to f$ ~~[illegible]~~ pointwise if

$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in X.$ (2)

book work

b) i) $\lim_{n\to\infty} f_n(0) = 1$

$\lim_{n\to\infty} f_n(x) = 0$ if $x > 0$ (2)

so f_n converges pointwise to

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Similar to examples and exercises

f is <u>not</u> in $C[0,\infty)$

OR

$f \notin C[0,\infty) \Rightarrow f_n$ does not converge to f uniformly. (This also follows from $f_n(\frac{1}{\sqrt{n}}) = e^{-1} > 0$.) (2)

EITHER ANSWER IS ACCEPTABLE

(ii) $\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \geq 0.$ (1)

$f_n \to f(x) = 0$ pointwise

$f_n'(x) = e^{-nx^2} - 2nx^2 e^{-nx^2}$

$= (1 - 2nx^2) e^{-nx^2}$

$= 0$ if $x = \frac{1}{\sqrt{2n}}$ (maximum)

$f_n(\frac{1}{\sqrt{2n}}) = \frac{1}{\sqrt{2n}} e^{-1/2} \to 0$ (2)

So $d(f_n, f) = \frac{1}{\sqrt{2n}} e^{-1/2} \to 0$ (1)

f_n converges to f uniformly.

book work

4. a) $A \subseteq X$ is compact if every sequence in A has a convergent subsequence (converging to an element of A). (3)

Similar to examples and exercises

b) Compact subsets of $\mathbb{R}$ are the closed, bounded, subsets.

(i) unbounded $\Rightarrow$ not compact (3)

(ii) closed, bounded $\Rightarrow$ compact (2)

(iii) contains the sequence $x_n = \frac{n+1}{n}$ which converges to $1 \notin A$

$\Rightarrow$ not closed $\Rightarrow$ not compact, (2) 5 a) Let $x_0 \in X$ and $\epsilon > 0$ be given. Choose any $0 < \delta < 1$. $d_X(x_1, x_0) < \delta$ $\Rightarrow x_1 = x_0 \Rightarrow$ $f(x_1) = f(x_0) \Rightarrow$ $d(f(x_1), f(x_0)) = 0 < \epsilon$

2

[illegible]

2

2

2

Unseen in this form. Similar to coursework and examples in lectures

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book work

b) $B_r(x) = \{y \in X : d(x,y) < r\}$ 2

$B_r[x] = \{y \in X : d(x,y) \leq r\}$ 2

c) Suppose $x_n \to x$ in X.

Let $\epsilon > 0$ be given.

$f^{-1}(B_\epsilon(f(x)))$ is open in X 3

and $x \in f^{-1}(B_\epsilon(f(x)))$ so 1

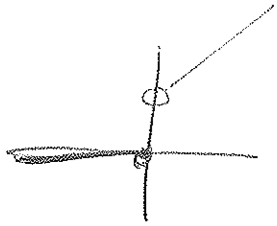
$\exists N > 0$ s.t. $n > N \Rightarrow x_n \in f^{-1}(B_\epsilon(f(x)))$. 3

Therefore, $n > N \Rightarrow f(x_n) \in B_\epsilon(f(x))$.

This implies $f(x_n) \to f(x)$. 3

done in lecture

a)



(not necessary to draw this)

Similar to examples and exercises

$S = f^{-1}((-\infty, \frac{1}{2})) = (-\infty, 0]$ 4

S is not open. 2

This implies f is not continuous, ~~as~~ because if f was continuous, the preimage of the open set $(-\infty, 1/2)$ would be open. 2

6. a) x_n is Cauchy if $\forall \epsilon > 0$ (3)

book work

$\exists N \geq 0$ such that $n, m > N \Rightarrow$

$d(x_n, x_m) < \epsilon$.

unseen but easy

b) i) No $\mathbb{Z}$ is unbounded (2)

ii) No $\mathbb{Q}$ is not closed. (2)

c.) e.g. $X = \mathbb{Q}$, $A = \{0\}$ (3)

~~c) Suppose $f_n \to f$ uniformly~~

c) i) Suppose f_n is a Cauchy sequence in $C[0,1]$ and $f_n(0) = 0$ $\forall n$.

$C[0,1]$ complete $\Rightarrow f_n \to f$ uniformly (2)

for some $f \in C[0,1]$. $f_n(0) = 0$ $\forall n$

$\Rightarrow f(0) = 0$, so f_n is ~~in the~~

converges to an element in the subspace (2)

of $C[0,1]$. This implies the

subspace is complete. (1)

unseen

Could be hard for them

ii) Let $f_n(x) = \frac{1}{n}$ $\forall x \in [0,1]$. (2)

$f_n \to f$, $f(x) = 0$ $\forall x \in [0,1]$, uniformly, (2)

but $f(0) = 0$ so f is not in

the subspace. This implies the

subspace is not complete. (1)

d) ~~Def~~ Suppose x_{n_k} is a subsequence and $x_{n_k} \to x$. Let $\epsilon > 0$ be given.

done in lecture

Let $N \geq 1$ be such that $d(x_n, x_m) < \frac{\epsilon}{2}$ whenever $n, m > N$. Choose $n_k > N$ large enough so that $d(x_{n_k}, x) > N$.

3

3

Then $n > N \Rightarrow d(x_n, x)$

$$
\begin{aligned}
&\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

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This implies $x_n \to x$.

7(a) i) Let $x_n = n$
Then no subsequence of x_n converges. (2)

Similar to examples and exercises

ii) Let $x_n = 1 - \frac{1}{n}$,
Then every subsequence of x_n converges to $1 \notin [0,1)$. (2)

iii) Let x_n be a sequence of rational numbers in $[0,1]$ converging to $\frac{\sqrt{2}}{2}$. (2)

done in lecture

b) Let X be compact and let $A \subseteq X$ be closed. Let x_n be a sequence in A. Since $A \subseteq X$ and X is compact, x_n has a subsequence $x_{n_k} \to x \in X$. Since x_{n_k} is in A and A is closed, $x \in A$. Thus A is compact (4) (4)

uncommented course work (with this proof given)

c) (i) If x_n is a sequence in $K \cap L$, x_n is a sequence in K, so $\exists$ a ~~s~~ subsequence $x_{n_k} \to x \in K$. Since x_{n_k} is a sequence in L, x_{n_k} has a subsequence $x_{n_{k_j}}$, convergent, and $x_{n_{k_j}} \to x \in L$. So $x \in K \cap L$. OR
Since L is compact, L is closed and $x \in L$. (6)

if x_n is a sequence in $K \cup L$, then either

$$|\{n \geq 1 : x_n \in K\}| = \infty \text{ or } \quad (*)$$
$$|\{n \geq 1 : x_n \in L\}| = \infty.$$

Uncommented

Course work (with this proof given)

Suppose WLOG the first case holds.
Let n_k be the k th element of $(*)$.

Since K is compact $\exists$ a ~~sub~~ convergent subsequence of $x_{n_k} \to x \in K \subseteq K \cup L$.
Thus, $K \cup L$ is compact.

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Unseen

(iii) Suppose x_n is a sequence in $K \cap L$.
Since x_n is in K, $\exists$ a convergent subsequence $x_{n_k} \to x \in K$. Since L is closed, $x \in L$.
So $x_{n_k} \to x \in K \cap L$ and $K \cap L$ is compact

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(ii) e.g. $[0,1] \cup \mathbb{R} = \mathbb{R}$ is not compact.

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