Queen Mary
University of London

B. Sc. Examination by course unit 2012

MTH6126 Metric Spaces

Duration: 2 hours

Date and time: 18th May 2012, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Dudley Stark

In this examination, $\mathbb{N} = \{1, 2, 3, \ldots\}$ stands for the set of natural numbers, $\mathbb{Z}$ stands for the set of integers, $\mathbb{Q}$ stands for the set of rational numbers, and $\mathbb{R}$ stands for the set of real numbers.

Section A: Each question carries 10 marks. You should attempt ALL FOUR questions.

Question 1 (a) Let X be a set and let $d : X \times X \to \mathbb{R}$ be a function. State the three axioms that d must satisfy in order to be a metric. [4]

(b) State the definitions of open and closed sets. [3]

(c) Prove that for $a < b$ in $\mathbb{R}$ with the Euclidean metric, the open interval (a, b) is an open set. [3]

Question 2 (a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Define what it means for f to be continuous. [3]

(b) Consider the metric spaces $(\mathbb{R}^3, d_1)$ and $(\mathbb{R}, d_1)$. Using the definition from (a), prove that the function $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = 2x + 3y + 4z$ is continuous. [7]

Question 3 (a) Let (f_n) be a sequence of functions between two metric spaces X and Y. Define what it means for the sequence f_n to converge to a function $f : X \to Y$ pointwise. [2]

(b) For each of the following sequences (f_n) of functions in $C[0, \infty)$ decide whether the sequence converges to a function f in $C[0, \infty)$ pointwise. If the sequence converges pointwise, determine whether the sequence converges to f uniformly.

(i) $f_n(x) = e^{-nx^2}$. [4]

(ii) $f_n(x) = xe^{-nx^2}$, [4]

Question 4 (a) Explain what it means for a subset $A \subseteq X$ of a metric space to be compact. [3]

(b) Which of the following sets A are compact, regarded as subspaces of $\mathbb{R}$ (with the Euclidean metric)?

(i) $A = \mathbb{Z}$ [3]

(ii) $A = \{-100, 200, 5000\}$ [2]

(iii) $A = \left\{\frac{n+1}{n} : n \in \mathbb{N}\right\}$ [2]

Briefly justify each of your answers. You may use a theorem proved in class characterizing the compact subsets of $\mathbb{R}$.

Section B: Each question carries 30 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 5 (a) Suppose that (X, d_X) and (Y, d_Y) are metric spaces where

$$d_X(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \neq x_2; \\ 0 & \text{if } x_1 = x_2. \end{cases}$$

Using the definition of continuous functions, prove that any function $f : X \to Y$ is continuous. [8]

(b) Given a metric space (X, d) and $x \in X$, $r > 0$, define what is meant by the open and closed balls $B_r(x)$ and $B_r[x]$. [4]

(c) Let $f : X \to Y$ be a continuous function between two metric spaces (X, d_X) and (Y, d_Y). If (x_n) is a convergent sequence in X, prove that $(f(x_n))$ is a convergent sequence in Y. You may assume that $f^{-1}(B_r(y))$ is open for any $y \in Y$, $r > 0$. [10]

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0; \\ x+1 & \text{if } x > 0. \end{cases}$$

Describe the inverse image $S = f^{-1}((-\infty, 1/2))$. Is the set S open in $\mathbb{R}$ with the Euclidean metric? Does your answer imply that f is either continuous or discontinuous? [8]

Question 6 (a) State what it means for a sequence in a metric space to be Cauchy. [3]

(b) Which of the following sets A are complete, regarded as subspaces of $\mathbb{R}$ (with the usual metric).

(i) $A = \mathbb{Z}$ [2]

(ii) $A = \mathbb{Q}$ [2]

Briefly justify your answers.

(c) State an example of metric space X and a subspace $A \subseteq X$, where X is not complete and A is complete. [3]

(d) Consider the complete metric space $C[0, 1]$ with the sup-metric

$$d(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|.$$

(i) Prove that $\{f \in C[0, 1] : f(0) = 0\}$ is complete as a subspace of $C[0, 1]$. [5]

(ii) Prove that $\{f \in C[0, 1] : f(0) > 0\}$ is not complete as a subspace of $C[0, 1]$. [5]

(e) Prove that if (X, d) is a metric space and if (x_n) is a Cauchy sequence in X with a convergent subsequence, then (x_n) is convergent. [10]

Question 7 (a) From first principles — i.e., directly from the definition you gave in Question 4 (a) — prove that the following subsets of $\mathbb{R}$ are not compact (with the usual metric).

(i) $[0, \infty)$. [2]

(ii) $[0, 1)$, [2]

(iii) $\mathbb{Q} \cap [0, 1]$. [2]

(b) Prove that any closed subset of a compact metric space is compact. [8]

(c) Let K, L be compact subsets of a metric space.

(i) Let K, L be compact subsets of a metric space. Prove that $K \cup L$ and $K \cap L$ are compact. [8]

(ii) Let K be a compact subset and let L be a closed subset of a metric space. Prove that $K \cap L$ is compact, but that $K \cup L$ need not be compact. [8]

End of Paper