Queen Mary
University of London

B. Sc. Examination by course unit 2011

MTH6126 Metric Spaces

Duration: 2 hours

Date and time: 9th May 2011, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Mark Jerrum

Section A: Each question carries 10 marks. You should attempt ALL FOUR questions.

Question 1 Suppose ϱ is a function $\varrho : X \times X \to \mathbb{R}$. Write down the axioms that must be satisfied by ϱ for it to be a *metric* on X.

A function $f : \{a, b, c\} \times \{a, b, c\} \to \mathbb{R}$ may be represented as a table. For example, the table

f	a	b	c
a	0	1	1
b	2	0	1
c	1	3	2

denotes the function f with $f(a, a) = 0$, $f(a, b) = 1$, $\ldots$, $f(c, c) = 2$.

Which of the following functions f_1, f_2, f_3 and f_4 are metrics on $\{a, b, c\}$? For each function that is not a metric, identify which axiom is violated.

f_1	a	b	c
a	0	1	2
b	2	0	1
c	2	1	0

f_2	a	b	c
a	0	1	2
b	1	0	1
c	2	1	1

f_3	a	b	c
a	0	1	3
b	1	0	2
c	3	2	0

f_4	a	b	c
a	0	1	3
b	1	0	1
c	3	1	0

Question 2 (a) Let (X, ϱ) and (Y, σ) be metric spaces, and f be a function from (X, ϱ) to (Y, σ). Explain what it means for f to be *continuous*.

(b) Assume that $\mathbb{R}$ is equipped with its usual metric, and the two element set $\{a, b\}$ with the discrete metric.

(i) Describe the continuous functions $\mathbb{R} \to \{a, b\}$.

(ii) Describe the continuous functions $\{a, b\} \to \mathbb{R}$.

Question 3 Denote by $C[0, \pi]$ the set of all real continuous functions on the closed interval $[0, \pi]$.

(a) Define the sup (or uniform) metric that makes $C[0, \pi]$ into a metric space.

(b) For each of the following sequences (f_n) of functions in $C[0, \pi]$ decide whether the sequence converges in $C[0, \pi]$. For the sequences that converge, state the limit function f. (No explanation is required.)

(i) $f_n(x) = \sin(nx)$,

(ii) $f_n(x) = n \sin(x/n)$,

(iii) $f_n(x) = (\frac{1}{2} \sin x)^n$ and

(iv) $f_n(x) = (\sin x)^n$.

Question 4 (a) Suppose that (x_n) is a sequence in the metric space (X, ϱ). Explain what it means for (x_n) to be a *Cauchy* sequence.

(b) Explain what it means for (X, ϱ) to be *complete*.

(c) Which of the following sets A are complete, regarded as subspaces of $\mathbb{R}$ (with the usual metric)?

(i) $A = \{0, 1, 2\}$,
(ii) $A = \{1/n : n \in \mathbb{Z} \setminus \{0\}\}$ and
(iii) $A = \{1/n : n \in \mathbb{Z} \setminus \{0\}\} \cup \{0\}$.

Briefly justify each of your answers. You may assume $\mathbb{R}$ is complete.

Section B: Each question carries 30 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 5 (a) Suppose (X, ϱ) is a metric space. Define the *open ball* $B_r(\alpha)$, where $\alpha \in X$ and $r > 0$. Explain what it means for a set $S \subseteq X$ to be *open*. [6]

(b) Let Ω be an arbitrary index set and $\{A_\omega : \omega \in \Omega\}$ be a collection of open sets in (X, ϱ) indexed by Ω. Prove that $S = \bigcup_{\omega \in \Omega} A_\omega$ is open. Prove that the intersection $A_1 \cap A_2$ of two open sets A_1, A_2 is open. [8]

(c) Consider the metric d^* on $\mathbb{R}^2$ defined by

$$d^*(p, q) = \begin{cases} |p_1 - q_1|, & \text{if } p_2 = q_2; \\ |p_1 - q_1| + 1, & \text{otherwise,} \end{cases}$$

where $p = (p_1, p_2)$ and $q = (q_1, q_2)$. Prove that, for any $\omega \in \mathbb{R}$, the set

$$A_\omega = (-2, 2) \times \{\omega\} = \{(x, \omega) : x \in (-2, 2)\}$$

is open in the metric space $(\mathbb{R}^2, d^*)$. [8]

(d) Deduce that $S = (-2, 2) \times [-2, 2]$ is an open set in $(\mathbb{R}^2, d^*)$. [3]

(e) Provide an example of an infinite collection of open sets $\{A_n : n \in \mathbb{N}\}$ in the metric space $(\mathbb{R}^2, d^*)$, whose intersection $S = \bigcap_{n \in \mathbb{N}} A_n$ is *not* open. You should state what S is, but you are not required to demonstrate that it is not open. [5]

Question 6 Suppose (X, ϱ) and (Y, σ) are metric spaces and f is a function from X to Y.

(a) Define the inverse image $f^{-1}(A)$ of a set $A \subseteq Y$. [4]

(b) Suppose that, for every open set $A \subseteq Y$, the inverse image $f^{-1}(A)$ of A is open. Prove that f is continuous (in the ε-δ sense). [10]

[*Hint.* Given $x \in X$ and $\varepsilon > 0$, consider an open ball $B_\varepsilon(y) \subseteq Y$ of radius ε centered at $y = f(x)$.]

(c) Consider the (discontinuous) function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 0, & \text{if } x = y = 0; \\ xy/(x^2+y^2), & \text{otherwise.} \end{cases}$$

(i) Describe the inverse image $S = f^{-1}(\mathbb{R} \setminus \{0\})$ of $\mathbb{R} \setminus \{0\}$ (i.e., the real line with the origin removed) under f. Is the set S open in $\mathbb{R}^2$ with the Euclidean metric? Briefly justify your answer. [8]

(ii) Describe the inverse image $S = f^{-1}((-\infty, \frac{1}{2}))$ of the open interval $(-\infty, \frac{1}{2})$. Is the set S open in $\mathbb{R}^2$ with the Euclidean metric? Briefly justify your answer. [8]

Question 7 (a) Let (X, ϱ) be a metric space, and K be a subset of X. Explain what it means for K to be (sequentially) compact. [4]

(b) Suppose $[a, b]$ is a closed interval of $\mathbb{R}$ with the usual metric, and suppose (x_n) is a sequence in $[a, b]$. Present a bisection procedure for constructing a subsequence of (x_n) that is a Cauchy sequence in $[a, b]$. [12]

(c) Does the procedure you described in part (b) continue to work when $[a, b]$ is replaced with $[a, b] \cap \mathbb{Q}$? If it doesn't, what goes wrong? If it does, discuss whether it follows that $[a, b] \cap \mathbb{Q}$ is compact (as a subset of $\mathbb{R}$). [4]

(d) Explain what it means for a subset $S \subseteq X$ of a metric space to be *bounded*. State a theorem relating the concepts *compact*, *closed* and *bounded* in the context of $\mathbb{R}^n$ with the Euclidean metric. [5]

(e) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $A \subseteq \mathbb{R}$. Which of the following statements are true in general and which false? For the ones that are false, provide a counterexample. [5]

(i) If A is compact then $f(A)$ is compact.

(ii) If A is closed then $f(A)$ is closed.

(iii) If A is bounded then $f(A)$ is bounded.

End of Paper