MTH6126 MAY 2011 EXAMINATION: SPECIMEN SOLUTIONS

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Q1 [Basic definition plus easy examples/counterexamples.] For all $x, y, z \in X$: (M1) $\varrho(x, y) \geq 0$ with equality iff $x = y$; (M2) $\varrho(x, y) = \varrho(y, x)$, and (M3) $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$. [1 mark each.]

f_1 is not a metric as it violates M2 ($f(a, b) \neq f(b, a)$); f_2 is not a metric as it violates M1 ($f(c, c) \neq 0$); f_3 is a metric; and f_4 is not a metric as it violates M3 ($f(a, c) \not\leq f(a, b) + f(b, c)$). [1 mark for each yes/no, and 1 additional mark for the violated axiom.]

Q2 [Basic definition plus easy examples.]

(a) E.g., f is continuous at $\alpha \in \mathbb{R}$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B^{\varrho}_{\delta}(\alpha)) \subseteq B^{\sigma}_{\varepsilon}(f(\alpha))$. ($B^{\varrho}_{\delta}(\alpha)$) is the ball in (X, ϱ) of radius δ centered at α, etc.) f is continuous if it is continuous at all $\alpha \in X$. (Definition in terms of images of convergent sequences also fine.) [4 marks.]

(b) (i) The only continuous functions are the constant function a and the constant function b. [3 marks.]

(ii) Any function from $\{a, b\}$ to $\mathbb{R}$ is continuous. [3 marks.]

Q3 (a) [Basic definition.] $d_\infty(f, g) = \sup_{x\in[0,\pi]} |f(x) - g(x)|$. [4 marks.]

(b) [(ii) and (iv) are from coursework; (i) and (iii) unseen.]

(i) Does not converge.

(ii) Converges to the identity function.

(iii) Converges to the zero function.

(iv) Does not converge. (Converges pointwise only.)

[1 mark for each yes/no, and 1 additional mark for the limit function.]

Q4 (a) [Basic definitions in the first two parts.] A sequence (x_n) is *Cauchy* if, for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $\varrho(x_n, x_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$. [2 marks.]

(b) (X, ϱ) is *complete* if every Cauchy sequence in (X, ϱ) converges to a point in X. [2 marks.]

(c) [Something similar to (ii) and (iii) in coursework.]

(i) Complete. (A is a closed subset of a complete space $\mathbb{R}$.)

(ii) Not complete. (The sequence $x_n = 1/n$ is a Cauchy sequence that converges in $\mathbb{R}$ to 0, but $0 \notin A$.)

(iii) Complete. (The complement of A is a union of open intervals and hence open. Thus A a closed subset of $\mathbb{R}$).

[1 mark for each yes/no answer, and an additional 1 mark for the explanation.]

Date: 31st December 2010.

Q5 (a) [Basic definition.] $B_r(\alpha) = \{x : \varrho(x,\alpha) < r\}$. [3 marks.] A set S is open if, for all $x \in S$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$. [3 marks.]

(b) [Bookwork.] Let $x \in S$ be arbitrary. Then $x \in A_\omega$ for some $\omega \in \Omega$. As A_ω is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_\omega \subseteq S$. Hence S is open. [4 marks.]

Let $x \in A_1 \cap A_2$ be arbitrary. Since A_1 is open, there is a ball $B_{r_1}(x)$ contained in A_1; similarly, there is a ball $B_{r_2}(x)$ contained in A_2. Let $r = \min\{r_1, r_2\}$; then the ball $B_r(x)$ is contained in $A_1 \cap A_2$. [4 marks.]

(c) [This metric appeared in coursework.] For any $a \in \mathbb{R}$ the ball $B_1((a,\omega))$ in $(\mathbb{R}^2, d^*)$ is the set $(a-1, a+1) \times \{\omega\}$. (A point p is in $B_1((a,\omega))$ iff $d^*((a,\omega),(p_1,p_2) < 1$. From the definition of d^*, the letter condition holds iff $p_2 = \omega$ and $|p_1 - a| < 1$.) Then A_ω is the union of open balls $(-2,0)\times\{\omega\}$, $(-1,1)\times\{\omega\}$ and $(0,2)\times\{\omega\}$, and hence open. [4 marks for saying what an open ball in the metric looks like, and 4 marks for completing the job.]

(d) [Easy consequence of (b) & (c).]

$$(-2,2) \times [-2,2] = \bigcup_{\omega \in [-2,2]} A_\omega.$$

The set in question is the union of open sets and hence open. [3 marks.]

(e) [Easy adaptation of a standard example from $\mathbb{R}$.] E.g., $A_n = (-1-1/n, 1+1/n) \times \{0\}$. For every n this set is open, by part (c). The intersection $\bigcap_{n\in\mathbb{N}} A_n$ is the set $S = [-1,1] \times \{0\}$. This set is not open. Consider the point $(1,0) \in S$. For all $\varepsilon > 0$ the ball $B_\varepsilon((1,0))$ is not contained in S: e.g., the point $(1+\varepsilon/2, 0)$ is in the ball but not in S. (Justification not required.) [5 marks.]

Q6 (a) [Basic definition.] $f^{-1}(A) = \{x \in X : f(x) \in A\}$. [4 marks.]

(b) [Bookwork.] Assume that the inverse image of any open set is open. Let $x \in X$, $\varepsilon > 0$ be arbitrary, and let $B_\varepsilon(y)$ be an open ball centred at $y = f(x) \in Y$ of radius ε. The inverse image $f^{-1}(B_\varepsilon(y))$ of the open ball is an open set [3 marks] that contains the point x (since $y = f(x)$ is certainly a member of $B_\varepsilon(y)$) [2 marks]. Therefore there exists a ball $B_\delta(x)$ about x such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(y))$ [3 marks] or, equivalently, $f(B_\delta(x)) \subseteq B_\varepsilon(y)$ [2 marks]. Since x, ε were arbitrary, it follows that f is continuous.

(c) [Students have seen this function before. Part (i) is unseen, part (ii) is from coursework.]

(i) If $x = 0$ or $y = 0$ then $f(x,y) = 0$; otherwise $f(x,y) \neq 0$. Thus $S = f^{-1}(\mathbb{R} \setminus \{0\})$ contains every point of $\mathbb{R}^2$ except the axes, i.e., $S = \{(x,y) : x \neq 0 \text{ and } y \neq 0\}$. The set S is open since, for every point $(x,y) \in S$, the ball $B_\varepsilon((x,y))$ is contained in S, where $\varepsilon = \min\{|x|,|y|\}$. [4 marks for description of inverse image, 2 for open/not open, 2 marks for justification.]

(ii) $S = \{(x, y) : f(x, y) < \frac{1}{2}\}$. Suppose $(x, y) \neq (0, 0)$. Then

$$\begin{aligned} f(x,y) < \tfrac{1}{2} &\iff xy/(x^2+y^2) < \tfrac{1}{2} \\ &\iff x^2 + y^2 - 2xy > 0 \\ &\iff (x-y)^2 > 0 \\ &\iff x \neq y. \end{aligned}$$

Also, $f(0, 0) = 0 < \frac{1}{2}$, so $(0, 0) \in S$. Summarising, $S = \{(x, y) : x \neq y \vee x = y = 0\}$. This set is not open, since the ball $B_\varepsilon((0, 0))$ is contained in the set for no $\varepsilon > 0$. [4 marks for description of inverse image, 2 for open/not open, 2 marks for justification.]

Q7 (a) [Basic definition.] K is *compact* if every sequence of points from K has a subsequence converging to a limit in K.

(b) [Bookwork.] Let (x_n) be an arbitrary sequence of numbers lying in a closed interval $[a, b]$. Let us split $[a, b]$ into the union of two intervals $[a, (a + b)/2]$ and $[(a + b)/2, b]$ of length $L/2$, where $L = b - a$ [3 marks]. At least one of these intervals contains infinitely many elements x_n of our sequence [2 marks]. Let us choose one of these elements, say x_{n_1}, and denote it by y_1 [2 marks]. Now we split the interval of length $L/2$ which contains infinitely many elements x_n into the union of two intervals of length $L/4$. Again, at least one of these intervals contains infinitely many elements of x_n. We choose one of these elements, say x_{n_2}, taking care that $n_2 > n_1$ [2 marks], and denote it by y_2. Repeating this procedure, we obtain a subsequence (y_k) of the sequence x_n such that y_k lies in an interval of length $2^{-N}L$ for all $k \geq N$. Clearly, (y_k) is a Cauchy sequence [3 marks].

(c) [Thought part.] The procedure works just as well for $[a, b] \cap \mathbb{Q}$, but it does not follow the set is compact. (Since $[a, b] \cap \mathbb{Q}$ is not complete, the Cauchy sequence does not necessarily converge to a point in $[a, b] \cap \mathbb{Q}$.) [2 marks for stating whether it works or not, 2 marks for completing the job.]

(d) [Basic definition/result.] The set S is *bounded* if there exists an element x and a number $r > 0$ such that $S \subseteq B_r(x)$. A subset of $\mathbb{R}^n$ (with Euclidean metric) is compact iff it is closed and bounded. [2 marks for definition of bounded, 3 marks for relevant result.]

(e) [(i) standard result; (ii) and (iii) unseen.] (i) True [1 mark], (ii) false [1 mark] (e.g., $A = [0, \infty)$ and $f(x) = 1/(1+|x|)$) [1 mark], and (iii) true [2 marks] (A is contained in a closed interval; f is continuous and hence bounded on that interval); *alternatively* 2 marks for "false", provided it was accompanied by a counterexample in which f is continuous *on* A. (I had intended f to be continuous just on A.)