Queen Mary
University of London

B. Sc. Examination by course unit 2010

MTH6126 Metric Spaces

Duration: 2 hours

Date and time: 6th May 2010, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Mark Jerrum

Section A: Each question carries 10 marks. You should attempt ALL FOUR questions.

Question 1 Explain what it means for sequence (s_n) to *converge to a point* $\alpha \in X$ in a metric space (X, ϱ), and what it means for metrics ϱ and σ on X to be *equivalent*.

Consider the function $d^* : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d^*(a,b) = \begin{cases} |a_1 - b_1|, & \text{if } a_2 = b_2; \\ |a_1 - b_1| + 1, & \text{otherwise,} \end{cases}$$

where $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Denote by d_2 the Euclidean metric on $\mathbb{R}^2$. Demonstrate that d^* and d_2 are *not* equivalent metrics. (You are not expected to prove that d^* is a metric.)

Question 2 Suppose (X, ϱ) is a metric space and $S \subseteq X$. Explain what it means for S to be *open*.

Recall that $B(0,1)$ is the space of bounded real functions on the open interval $(0,1)$, with the sup (or uniform) metric. Which of the following sets in $B(0,1)$ are open, and why?

(a) $S = \{f \in B(0,1) : \sup_{x\in(0,1)} |f(x)| \leq 1\}$,

(b) $S = \{f \in B(0,1) : \sup_{x\in(0,1)} |f(x)| < 1\}$, and

(c) $S = \{f \in B(0,1) : |f(x)| < 1, \text{ for all } x \in (0,1)\}$. [Hint: consider the identity function on $(0,1)$.]

Question 3 Suppose $f : (X, \varrho) \to (Y, \sigma)$ is a mapping from one metric space to another. Give a criterion for f to be continuous at $\alpha \in X$ in terms of convergent sequences in X.

By exhibiting a suitable sequence in $\mathbb{R}^2$, demonstrate that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 0, & \text{if } (x,y) = (0,0); \\ 2xy/(x^2+y^2), & \text{otherwise,} \end{cases}$$

is discontinuous at $(0,0)$. (Assume that $\mathbb{R}^2$ and $\mathbb{R}$ are equipped with the Euclidean metric.)

Suppose we modify the definition of f so that $f(0,0) = c$. Is there any choice for c that makes f continuous at $(0,0)$?

Question 4 Explain what it means for a set S in a metric space (X, ϱ) to be *compact*.

Which of the following subsets of $\mathbb{R}^2$ with the Euclidean metric are compact?

(a) $S = \{(x,y) : x^2 + y^2 \leq 1\}$,

(b) $S = \{(x,y) : x^2 + y^2 < 1\}$,

(c) $S = \{(x,y) : y \geq x^2\}$, and

(d) $S = \{(x,y) : y > x^2\}$.

Briefly justify your answers.

Section B: Each question carries 30 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 5 Throughout this question, (X, ϱ) is a metric space.

(a) Suppose $\alpha \in X$ and $r > 0$. Define the open ball $B_r(\alpha)$ in (X, ϱ), and prove that it is an open set. [6]

(b) Let Ω be an arbitrary index set and $\{A_\omega : \omega \in \Omega\}$ a collection of open sets in (X, ϱ) indexed by Ω. Prove that $S = \bigcup_{\omega \in \Omega} A_\omega$ is open. [6]

(c) Prove that a set A in (X, ϱ) is open if and only if it can be expressed as a union of a collection of open balls. [5]

(d) Give a definition of the *closure* $\overline{A}$ of a set $A \subset X$. [3]

(e) Define A^* to be the intersection of all closed balls containing A, that is

$$A^* = \bigcap \{B_r[x] : r \in \mathbb{R}^+, x \in X \text{ and } B_r[x] \supseteq A\}.$$

Prove that $\overline{A} \subseteq A^*$. [5]

(f) Now specialise (X, ϱ) to the space $\mathbb{R}$ with the usual metric. Present a simple example to demonstrate that the inclusion in part (e) is strict. [5]

Question 6 Recall that $B[0, \pi]$ is the space of bounded real functions on $[0, \pi]$ equipped with the sup (or uniform) metric d_∞. In this question, *uniform* convergence will mean convergence in the metric space $B[0, \pi]$.

(a) Suppose (f_n) is a sequence of functions in $B[0, \pi]$. Explain what it means for the sequence (f_n) to converge *pointwise* to a function $f \in B[0, \pi]$. State a relationship that holds between pointwise and uniform convergence. [6]

(b) Demonstrate that the sequence $(f_1, f_2, \ldots)$ defined by $f_n(x) = \sin(x/n)$ converges uniformly to some $f \in B[0, \pi]$, and determine f. [Note: By the Mean Value Theorem, $|\sin z| \leq |z|$ for all $z \in \mathbb{R}$.] [5]

(c) Demonstrate that the sequence $f_n(x) = (\sin x)^n$ converges pointwise to some function f, and determine f. [6]

(d) Suppose (f_n) is a sequence of continuous functions in $B[0, \pi]$ converging (uniformly) to a function f. Prove that f is continuous. [Hint: Given $\varepsilon > 0$ consider a function f_n in the sequence satisfying $d_\infty(f_n, f) < \varepsilon/3$.] [10]

(e) Deduce that the convergence of the sequence in part (c) is not uniform. [3]

Question 7 Throughout the question, (X, ϱ) and (Y, σ) are metric spaces.

(a) Explain what it means for a sequence (x_n) in (X, ϱ) to be *Cauchy*, and what it means for (X, ϱ) to be *complete*. [6]

(b) Suppose that the metric space (X, ϱ) is complete. Prove that if $A \subseteq X$ is closed in (X, ϱ) then the subspace (A, ϱ) is complete. [6]

(c) Which of the following sets $A \subseteq \mathbb{R}$ are complete as subspaces of $\mathbb{R}$ with the usual metric? Justify your answers. [6]

(i) $A = [0, 1]$,

(ii) $A = \{2^n : n \in \mathbb{N}\}$, and

(iii) $A = \{2^{-n} : n \in \mathbb{N}\}$.

(d) Suppose (X, ϱ) is complete and that f is a continuous function from (X, ϱ) to (Y, σ). Is it necessarily the case that $f(X)$, the image of X under f, is complete (as a subspace of (Y, σ))? Justify your answer. [6]

(e) Now repeat part (d), but with (X, ϱ) specialised to the closed interval $[0, 1]$ with the usual metric on $\mathbb{R}$. Is $f([0, 1])$ necessarily complete? Again, justify your answer. [6]

End of Paper