MTH6126 MAY 2010 EXAMINATION: SPECIMEN SOLUTIONS

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(1) In addition to ε/δ, I would accept: (s_n) converges to α iff $\varrho(s_n, \alpha) \to 0$ as $n \to \infty$. Metrics ϱ and σ are equivalent iff every sequence converging to some point α in (X, ϱ) also converges to α in (Y, σ), and vice versa. [Standard definitions.]

Consider the sequence (s_n) given by $s_n = (0, 1/n)$. Note that $d_2(s_n, (0,0)) = 1/n$ but $d^*(s_n, (0,0)) = 1$, so (s_n) converges to $(0,0)$ in $(\mathbb{R}^2, d_2)$ but not in $(\mathbb{R}_2, d^*)$. [Students have seen the metric d^* before, if not this question.]

(2) S is open iff for all $x \in X$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$. [Standard definition.]

(a) Not open. Let $c_1 : (0,1) \to \mathbb{R}$ be the constant function 1. Then $c_1 \in S$, since $\sup_{x\in(0,1)} |c_1(x)| = 1$. However, for all $\varepsilon > 0$, the open ball $B_\varepsilon(c_1)$ centred at c_1 contains the constant function $1 + \varepsilon/2$, which is outside S.

(b) Open. Suppose $f \in S$ and let $\varepsilon = 1 - \sup_{x\in(0,1)} |f(x)| > 0$. Then the ball $B_\varepsilon(f)$ is contained in S. To see this, note that if $g \in B_\varepsilon(f)$, then

$$\sup_{x\in(0,1)} |g(x)| \leq \sup_{x\in(0,1)} (|g(x) - f(x)| + |f(x)|) < \varepsilon + (1 - \varepsilon) = 1.$$

(c) Not open. Consider the identity function $I(x) = x$. Then $\sup_{x\in(0,1)} |I(x)| = 1$, and we can argue as in (a).

[These are unseen, but straightforward, except possibly (c).]

(3) The function f is continuous at α iff, for every sequence (x_n) converging to α in (X, ϱ), the sequence $f(x_n)$ converges to $f(\alpha)$ in (Y, σ). [Bookwork.]

Consider the sequence $s_n = (1/n, 1/n)$, which converges to $(0,0)$ in $\mathbb{R}^2$ with the Euclidean metric. Observe that $f(s_n) = f(1/n, 1/n) = 1$, so $f(s_n) \to 1 \neq 0 = f(0,0)$ in $\mathbb{R}$. [Students have seen this example, without the factor 2, in the course.]

No. (The sequence $t_n = (1/n, 0)$ converges to $(0,0)$ in $\mathbb{R}^2$, and $f(t_n) = 0$ converges to 0 in $\mathbb{R}$. So one of (s_n) or (t_n) will always be a counterexample.) [Unseen, but implicit in the discussion of the example in the course.]

(4) S is compact iff every sequence in S has a convergent subsequence. [Standard definition.]

(a) Compact. It is a closed, bounded subset of $\mathbb{R}^2$ and hence compact by Heine-Borel.

(b) Not compact. The sequence $s_n = (1 - 1/n, 0)$ in S converges to $(1, 0) \notin S$, and so does any subsequence.

(c) Not compact. The sequence $s_n = (0, n)$ has the property that $d_2(s_n, s_m) \geq 1$ for all n, m with $n \neq m$. So no subsequence of (s_n) is Cauchy, and hence no subsequence is convergent.

Date: 31st December 2009.

(d) Not compact. As for (c).

[I'm not aware the students have seen exactly these examples, but they have seen similar.]

(5) (a) $B_r(\alpha) = \{x : \varrho(\alpha, x) < r\}$. Suppose $x \in B_r(\alpha)$. Then $\varrho(\alpha, x) = r - \varepsilon$ for some $\varepsilon > 0$. Consider the ball $B_\varepsilon(x)$, and let $y \in B_\varepsilon(x)$ be arbitrary. By the triangle inequality, $\varrho(\alpha, y) \leq \varrho(\alpha, x) + \varrho(x, y) < 1 - \varepsilon + \varepsilon = 1$. So $y \in B_r(\alpha)$. But y was an arbitrary point in $B_\varepsilon(x)$. [Bookwork.]

(b) Let $x \in S$ be arbitrary. There exists $\omega \in \Omega$ with $x \in A_\omega$. Since A_ω is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_\omega \subseteq S$. But $x \in S$ was arbitrary, so S is open. [Bookwork.]

(c) Suppose A is open. For each $x \in A$ choose $\varepsilon(x)$ so that $B_{\varepsilon(x)}(x) \subseteq A$. Clearly, $A = \bigcup_{x \in X} B_{\varepsilon(x)}(x)$. Conversely, the union of any collection of open balls is open by (a) and (b). [Bookwork.]

(d) $\overline{A}$ is the smallest (with respect to set inclusion) closed set containing A. [Standard definition.]

(e) A^* is an intersection of a collection of closed sets and hence closed. It clearly contains A, and hence contains the smallest closed set containing A. [Unseen.]

(f) $A = \{0, 1\}$. $\overline{A} = A = \{0, 1\}$ since A is closed, whereas $A^* = [0, 1]$. [Unseen.]

(6) (a) For all $x \in [0, \pi]$, for all $\varepsilon > 0$, there exists $N_{x,\varepsilon}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N_{x,\varepsilon}$. Uniform convergence implies pointwise convergence. [Standard definition.]

(b) f is the constant 0 function. Then

$$\begin{aligned} d_\infty(f_n, f) &= \sup_{x \in [0,\pi]} |f_n(x) - f(x)| = \sup_{x \in [0,\pi]} |\sin(x/n)| \\ &\leq \sup_{x \in [0,\pi]} |x/n| = \pi/n. \end{aligned}$$

So $d_\infty(f_n, f) \to 0$ as $n \to \infty$. [Appeared in coursework.]

(c) $f(x) = 1$ if $x = \pi/2$; $f(x) = 0$ otherwise. When $x = \pi/2$, $f_n(x) = (\sin(\pi/2))^n = 1^n = 1$. Hence $f_n(\pi/2) \to 1$ as $n \to \infty$. If $x \neq \pi/2$, $0 \leq f_n(x) < 1$ and so $f_n(x) = (\sin(\pi/2))^n \to 0$ as $n \to \infty$. [Something similar in coursework.]

(d) Suppose $\alpha \in [0, \pi]$ and $\varepsilon > 0$. Since $f_n \to f$ we can choose n such that $d_\infty(f_n, f) \leq \varepsilon/3$, i.e., $|f_n(x) - f(x)| \leq \varepsilon/3$ for all $x \in [0, \pi]$. Since f_n is continuous, there exists $\delta > 0$ such that $|f_n(x) - f_n(\alpha)| < \varepsilon/3$ whenever $|x - \alpha| < \delta$. So, by the triangle inequality,

$$\begin{aligned} |f(x) - f(\alpha)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(\alpha)| + |f_n(\alpha) - f(\alpha)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

whenever $|x - \alpha| < \delta$. [Bookwork.]

(e) f_n are all continuous. If convergence was uniform then the limit would be continuous also, but it is not. [Simple observation.]

(7) (a) (x_n) is *Cauchy* iff for all $\varepsilon > 0$ there exists N_ε such that $\varrho(x_n, x_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$. (X, ϱ) is *complete* if every Cauchy sequence (x_n) converges to a limit in X. [Standard definitions.]

(b) Suppose (x_n) is any Cauchy sequence in (A, ϱ). Since (x_n) is also a Cauchy sequence in (X, ϱ), and (X, ϱ) is complete, (x_n) converges to a limit in X, say $\alpha \in X$. We just need to show that $\alpha \in A$. Suppose not. The sequence (x_n) witnesses the fact that α is a limit point of A. But A is closed and hence contains all its limit points, a contradiction. [Bookwork.]

(c) (i) Complete. A is closed subset of the complete space $\mathbb{R}$, so is closed by (b).

(ii) Complete. Ditto. (Observe that the complement of A is a union of open intervals, and hence open. A itself is therefore closed.)

(iii) Not complete. The sequence x_n defined by $x_n = 2^{-n}$ is Cauchy, and converges to $0 \notin A$.

[Variations of examples that have appeared in the course.]

(d) No. $[1, \infty)$ is complete as a subspace of $\mathbb{R}$ with the usual metric; however its image under $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/x$ is $(0, 1]$ which is not complete. [Unseen.]

(e) Yes. $[0, 1]$ is compact and so its image under a continuous function is compact. A compact space is complete. [Unseen.]