Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNITS

MTH6126 Metric Spaces

Monday 27th April 2009, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

This examination paper may not be removed from the examination room.

Examiners: Mark Jerrum and Cho-Ho Chu.

SECTION A

This section carries 40 marks and each question carries 10 marks. You should attempt ALL FOUR questions.

A1. Denote by $B(S)$ the set of bounded real-valued functions on a set S. Define $\varrho : B(S)^2 \to \mathbb{R}$ by

$$\varrho(f,g) = \sup_{x \in S} |f(x) - g(x)|$$

for all $f, g \in B(S)$. Prove that ϱ satisfies the triangle inequality. Why is it necessary to assume that the functions in $B(S)$ are bounded? Provide an illustrative example to show that, without this assumption, ϱ may fail to be well-defined.

A2. Suppose (X, ϱ) is a metric space. Describe an *open ball* in (X, ϱ). Explain what it means for a set $A \subseteq X$ to be *open* and what it means for it to be *closed*.

Now specialise (X, ϱ) to be $\mathbb{R}$ with the usual metric. Denote by $A \oplus B$ the *symmetric difference* of sets $A, B \subseteq X$, that is to say $A \oplus B = (A \setminus B) \cup (B \setminus A)$. Give examples of open sets $\emptyset \subset A, B \subset \mathbb{R}$ such that (i) $A \oplus B$ is open but not closed, (ii) $A \oplus B$ is closed but not open, and (iii) $A \oplus B$ is neither open nor closed. Briefly explain each of your answers.

A3. Explain what it means for a metric space (X, ϱ) to be *complete*.

Which of the following subsets of $\mathbb{R}$ are complete when considered as subspaces of $\mathbb{R}$ with the usual metric?

(a) $(0, \infty)$,

(b) $[0, \infty)$,

(c) $\{n^{-2} : n = 1, 2, \ldots\}$,

(d) $\{n^{-2} : n = 1, 2, \ldots\} \cup \{0\}$, and

(e) $\mathbb{Q} \cap [0, 1]$.

Briefly justify each of your answers.

A4. Explain what it means for a subset K of a metric space (X, ρ) to be (sequentially) *compact*.

Demonstrate from first principles (i.e., directly from the definition) that neither $[0, \infty) \times [-1, 1]$ nor $(0, 1) \times [-1, 1]$ are compact subsets of $\mathbb{R}^2$ with the Euclidean metric.

State a standard theorem from which it may be deduced that $[0, 1] \times [-1, 1]$ *is* a compact subset of $\mathbb{R}^2$.

SECTION B

This section carries 60 marks and each question carries 30 marks. You may attempt all three questions but only marks for the BEST TWO questions will be counted.

B1. (a) [8 marks] Suppose (X, ϱ) is a metric space. Define $\sigma : X^2 \to \mathbb{R}$ by $\sigma(x, y) = \sqrt{\varrho(x, y)}$ for all $x, y \in X$. Prove that σ is a metric on X. [Hint: show that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, for all $a, b \geq 0$.]

(b) [5 marks] Define $\tau : X^2 \to \mathbb{R}$ by $\tau(x, y) = \varrho(x, y)^2$ for all $x, y \in X$. By presenting a counterexample with $|X| = 3$, demonstrate that τ may not be a metric on X.

(c) [6 marks] With ϱ, σ as in part (a), prove that a set $A \subseteq X$ is open in (X, σ) if it is open in (X, ϱ).

(d) [6 marks] Let $T : X \to X$ be any injective map from X to itself. Define $\varrho' : X^2 \to \mathbb{R}$ by $\varrho'(x, y) = \varrho(T(x), T(y))$. Prove that ϱ' is a metric on X.

(e) [5 marks] Are the open sets in (X, ϱ) and (X, ϱ') necessarily the same? Explain your answer.

B2. (a) [4 marks] State the condition for a mapping f of a metric space (X, ϱ) to itself to be a *contraction*.

(b) [4 marks] State the contraction mapping theorem.

(c) [12 marks] Denote by d_1 the Manhattan or ℓ_1 metric on $\mathbb{R}^2$; that is, $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Prove that $(\mathbb{R}^2, d_1)$ is a complete metric space. (You may assume that $\mathbb{R}$ is complete with the usual metric.)

(d) [10 marks] Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (\frac{1}{2}x_2, \frac{1}{2}(x_1 + 1))$ is a contraction on $(\mathbb{R}^2, d_1)$. What is the fixed point of f?

[Next question overleaf]

B3. Suppose (X, ϱ) and (Y, σ) are metric spaces, and $f : X \to Y$ is a map between them.

(a) [4 marks] Explain what it means for f to be *continuous*.

(b) [10 marks] Suppose $f : X \to Y$ is continuous. Let x_n be any sequence of points in X converging to $\alpha \in X$. Prove that $f(x_n)$ converges to $f(\alpha)$ in Y.

(c) [8 marks] Again suppose f is continuous, and further suppose K is a compact subset of X. Prove that the image $f(K)$ of K is compact.

(d) [8 marks] Let L be a subset of Y. Define the inverse image $f^{-1}(L)$ of L. Assume L is compact and f is continuous. Is $f^{-1}(L)$ necessarily closed? Is $f^{-1}(L)$ necessarily bounded? Is $f^{-1}(L)$ necessarily compact? Explain your answers.

 [End of examination paper]