Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNITS

MTH6126 Metric Spaces

Monday 27th April 2009, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

This examination paper may not be removed from the examination room.

Examiners: Mark Jerrum and Cho-Ho Chu.

SECTION A

This section carries 40 marks and each question carries 10 marks. You should attempt ALL FOUR questions.

A1. Denote by $B(S)$ the set of bounded real-valued functions on a set S. Define $\varrho : B(S)^2 \to \mathbb{R}$ by

$$\varrho(f, g) = \sup_{x \in S} |f(x) - g(x)|$$

for all $f, g \in B(S)$. Prove that ϱ satisfies the triangle inequality. Why is it necessary to assume that the functions in $B(S)$ are bounded? Provide an illustrative example to show that, without this assumption, ϱ may fail to be well-defined.

A2. Suppose (X, ϱ) is a metric space. Describe an *open ball* in (X, ϱ). Explain what it means for a set $A \subseteq X$ to be *open* and what it means for it to be *closed*.

Now specialise (X, ϱ) to be $\mathbb{R}$ with the usual metric. Denote by $A \oplus B$ the *symmetric difference* of sets $A, B \subseteq X$, that is to say $A \oplus B = (A \setminus B) \cup (B \setminus A)$. Give examples of open sets $\emptyset \subset A, B \subset \mathbb{R}$ such that (i) $A \oplus B$ is open but not closed, (ii) $A \oplus B$ is closed but not open, and (iii) $A \oplus B$ is neither open nor closed. Briefly explain each of your answers.

A3. Explain what it means for a metric space (X, ϱ) to be *complete.*

Which of the following subsets of $\mathbb{R}$ are complete when considered as subspaces of $\mathbb{R}$ with the usual metric?

(a) $(0, \infty)$,

(b) $[0, \infty)$,

(c) $\{n^{-2} : n = 1, 2, \ldots\}$,

(d) $\{n^{-2} : n = 1, 2, \ldots\} \cup \{0\}$, and

(e) $\mathbb{Q} \cap [0, 1]$.

Briefly justify each of your answers.

A4. Explain what it means for a subset K of a metric space (X, ρ) to be (sequentially) *compact.*

Demonstrate from first principles (i.e., directly from the definition) that neither $[0, \infty) \times [-1, 1]$ nor $(0, 1) \times [-1, 1]$ are compact subsets of $\mathbb{R}^2$ with the Euclidean metric.

State a standard theorem from which it may be deduced that $[0, 1] \times [-1, 1]$ *is* a compact subset of $\mathbb{R}^2$.

[Next section overleaf]

SECTION B

This section carries 60 marks and each question carries 30 marks. You may attempt all three questions but only marks for the BEST TWO questions will be counted.

B1. (a) [8 marks] Suppose (X, ϱ) is a metric space. Define $\sigma : X^2 \to \mathbb{R}$ by $\sigma(x, y) = \sqrt{\varrho(x, y)}$ for all $x, y \in X$. Prove that σ is a metric on X. [Hint: show that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$, for all $a, b \geq 0$.]

(b) [5 marks] Define $\tau : X^2 \to \mathbb{R}$ by $\tau(x, y) = \varrho(x, y)^2$ for all $x, y \in X$. By presenting a counterexample with $|X| = 3$, demonstrate that τ may not be a metric on X.

(c) [6 marks] With ϱ, σ as in part (a), prove that a set $A \subseteq X$ is open in (X, σ) if it is open in (X, ϱ).

(d) [6 marks] Let $T : X \to X$ be any injective map from X to itself. Define $\varrho' : X^2 \to \mathbb{R}$ by $\varrho'(x, y) = \varrho(T(x), T(y))$. Prove that ϱ' is a metric on X.

(e) [5 marks] Are the open sets in (X, ϱ) and (X, ϱ') necessarily the same? Explain your answer.

B2. (a) [4 marks] State the condition for a mapping f of a metric space (X, ϱ) to itself to be a *contraction*.

(b) [4 marks] State the contraction mapping theorem.

(c) [12 marks] Denote by d_1 the Manhattan or ℓ_1 metric on $\mathbb{R}^2$; that is, $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Prove that $(\mathbb{R}^2, d_1)$ is a complete metric space. (You may assume that $\mathbb{R}$ is complete with the usual metric.)

(d) [10 marks] Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (\frac{1}{2}x_2, \frac{1}{2}(x_1 + 1))$ is a contraction on $(\mathbb{R}^2, d_1)$. What is the fixed point of f?

[Next question overleaf]

B3. Suppose (X, ϱ) and (Y, σ) are metric spaces, and $f : X \to Y$ is a map between them.

(a) [4 marks] Explain what it means for f to be *continuous*.

(b) [10 marks] Suppose $f : X \to Y$ is continuous. Let x_n be any sequence of points in X converging to $\alpha \in X$. Prove that $f(x_n)$ converges to $f(\alpha)$ in Y.

(c) [8 marks] Again suppose f is continuous, and further suppose K is a compact subset of X. Prove that the image $f(K)$ of K is compact.

(d) [8 marks] Let L be a subset of Y. Define the inverse image $f^{-1}(L)$ of L. Assume L is compact and f is continuous. Is $f^{-1}(L)$ necessarily closed? Is $f^{-1}(L)$ necessarily bounded? Is $f^{-1}(L)$ necessarily compact? Explain your answers.

[Next section overleaf]

Specimen solutions

SECTION A

A1. Let $f, g, h \in B(S)$. For all $x \in S$, we have

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \leq \varrho(f,g) + \varrho(g,h).$$

So $\varrho(f,g) + \varrho(g,h)$ is an upper bound on $|f(x) - g(x)|$ for all $x \in S$, and hence is at least as great as the least upper bound $\sup_{x \in S} |f(x) - h(x)| = \varrho(f,h)$.

Boundedness ensures that the set $\{|f(x) - g(x)| : x \in S\}$ has a least upper bound. Suppose $S = (0,1)$, $f(x) = 1/x$ and $g(x) = 0$. Then $\sup_{x \in S} |f(x) - g(x)| = \infty$.

A2. An open ball in (X, ϱ) is a set of the form $B_r(\alpha) = \{x : \varrho(\alpha, x) < r\}$, for some $\alpha \in X$ and $r > 0$. A set $A \subseteq X$ is open if, for every $\alpha \in X$, there exists $\varepsilon > 0$ such that $B_\varepsilon(\alpha) \subseteq X$. A set A is closed if its complement $X \setminus A$ is open.

If $A = (0,1)$ $B = (2,3)$ then A, B are both open. Also, $A \oplus B = (0,1) \cup (2,3)$, which is open (union of open intervals) but not closed (0 is a limit point lying outside the set). If $A = (0, \infty)$ and $B = (0,1)$ then $A \oplus B = [1, \infty)$ which is closed but not open (contains no open ball centered at 1). If $A = (0,2)$ and $B = (0,1)$, then $A \oplus B = [1,2)$, which is not open (there is no ball centered at 1 and contained in $[1,2)$) and not closed (2 is a limit point not contained in $[1,2)$).

A3. (X, ϱ) is complete iff every Cauchy sequence in X converges to a point in X.

(a) No. The sequence $x_n = 1/n$ converges to 0 in $\mathbb{R}$, and $0 \notin (0,1]$.

(b) Yes. $[0, \infty)$ is a closed subset of $\mathbb{R}$, which is a complete metric space.

(c) No. Consider the subsequence $x_n = n^{-2}$ and argue as in (a).

(d) Yes. The complement of the set is $(-\infty, 0) \cup (1, \infty) \cup (2^{-2}, 1) \cup (3^{-2}, 2^{-2}) \cup (4^{-2}, 3^{-2}) \cup \cdots$, which is a union of open sets and hence open. Therefore the set itself is closed. Now argue as in (b).

(e) No. Consider a sequence of rational numbers in $[0,1]$ converging to $1/\sqrt{2}$.

A4. K is (sequentially) compact iff every sequence of elements in K contains a subsequence converging to a point in K.

[This question continues overleaf ...]

Consider the sequence $x_n = (n,0)$ in $[0,\infty)\times[-1,1]$. For all $n\neq m$, we have $\|x_n - x_m\| \geq 1$. So x_n contains no Cauchy subsequence and hence no convergent subsequence. Consider the sequence $x_n = (1/(n+1),0)$ in $(0,1)\times[-1,1]$. This sequence converges to the point $(0,0)$ in $\mathbb{R}^2$ and so too does any subsequence. But $(0,0)\notin(0,1)\times[-1,1]$.

Heine-Borel: a subset of R^n is compact iff it is closed and bounded.

SECTION B

B1. (a) [Unseen, though $\sigma(x,y)=\min\{\varrho(x,y),1\}$ was set as an exercise.]

- $\sigma(x,x)=\sqrt{\varrho(x,x)}=\sqrt{0}=0$; and $\sigma(x,y)=\sqrt{\varrho(x,y)}>0$ provided $x\neq y$.
- $\sigma(x,y)=\sqrt{\varrho(x,y)}=\sqrt{\varrho(y,x)}=\sigma(y,x)$.
- For any $a,b\geq 0$, we have $a+b\leq a+2\sqrt{ab}+b$. Taking square roots, $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ since $\sqrt{\cdot}$ is a monotonic function. Then $\sigma(x,y)+\sigma(y,z)=\sqrt{\varrho(x,y)}+\sqrt{\varrho(y,z)}\geq\sqrt{\varrho(x,y)+\varrho(y,z)}\geq\sqrt{\varrho(x,z)}=\sigma(x,z)$. The final inequality is by the triangle inequality for ϱ.

(b) [Set as coursework, without the constraint $|X|=3$.] $X=\{a,b,c\}$, $\varrho(a,b)=\varrho(b,c)=1$ and $\varrho(a,c)=2$. Then (X,ϱ) is a metric space, but (X,σ) fails the triangle inequality: $\tau(a,b)+\tau(b,c)=2<4=\tau(a,c)$.

(c) [Simple example of equivalent metrics, which is covered in the course.] For $r>0$, $x\in X$, denote by $B^{\varrho}_r(x)$ (resp. $B^{\sigma}_r(x)$) the ball in (X,ϱ) (resp. (X,σ)) of radius r centered at x. Let A be a set open in (X,ϱ). Take any point $x\in X$. For some $\varepsilon>0$ the ball $B^{\varrho}_{\varepsilon}(x)$ is contained in A. $B^{\varrho}_{\varepsilon}(x)=B^{\sigma}_{\delta}(x)$ where $\delta=\sqrt{\varepsilon}$. So there is a ball $B^{\sigma}_{\delta}(x)$ centred at x and contained in A. But $x\in A$ was chosen arbitrarily, and hence A is open in (X,σ).

(d) [Unseen.]

- $\varrho'(x,x)=\varrho(T(x),T(x))=0$; and $\varrho'(x,y)=\varrho(T(x),T(y))>0$ provided $x\neq y$ (since T is injective).
- $\varrho'(x,y)=\varrho(T(x),T(y))=\varrho(T(y),T(x))=\varrho'(y,x)$.
- $\varrho'(x,y)+\varrho'(y,z)=\varrho(T(x),T(y))+\varrho(T(y),T(z))\geq\varrho(T(x),T(z))=\varrho'(x,z)$. The final inequality is by the triangle inequality for ϱ.

(e) [Unseen.] No. E.g., (X,ϱ) is $\mathbb{R}$ with the usual metric, and $T(x)=x-1$ if $x<0$, $T(0)=0$, and $T(x)=x+1$ if $x>0$. Then $\{0\}$ is an open set in (X,ϱ') (indeed it is an open ball), but $\{0\}$ is not an open set in the usual metric.

B2. (a) [The first two parts are bookwork.] A map f from a metric space (X,ϱ) to itself is called a *contraction* if $\varrho(f(x),f(y))\leq c\varrho(x,y)$ for some $0\leq c<1$ and all $x,y\in X$.

[This question continues overleaf ...]

(b) If f is a contraction on a complete metric space then the equation $f(x) = x$ has a unique solution x and, for any $x_0 \in X$, the sequence x_n defined by $x_n = f(x_{n-1})$, for all $n > 0$, converges to x.

(c) [This was an exercise in the course, but with Euclidean metric in place of d_1.] Suppose (a_n, b_n) is a Cauchy sequence in $(\mathbb{R}^2, d_1)$. Then for all $\varepsilon > 0$ there exists N_ε such that $d_1((a_n, b_n), (a_m, b_m)) = |a_m - a_n| + |b_m - b_m| \le \varepsilon$ for all $n, m \ge N_\varepsilon$. Clearly, the sequences a_n and b_n are Cauchy in $\mathbb{R}$ (with the same N_ε).

Since $\mathbb{R}$ is complete, each of these sequences converges to a respective limit in $\mathbb{R}$, say α and β. We claim that (α, β) is the limit of the original sequence in $(\mathbb{R}^2, d_1)$. For any ε there exist A_ε and B_ε such that $|a_n - \alpha| < \varepsilon/2$ when $n \ge A_\varepsilon$ and $|b_n - \beta| < \varepsilon/2$ when $n \ge B_\varepsilon$. Then

$$d_1\big((a_n, b_n), (\alpha, \beta)\big) = |a_n - \alpha| + |b_n - \beta| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $n \ge \max\{A_\varepsilon, B_\varepsilon\}$. But $\varepsilon > 0$ was arbitary, so (a_n, b_n) converges to (α, β) in $(\mathbb{R}^2, d_1)$.

(d) [Fairly routine calculation.] We have

$$\begin{aligned} d_1(f(x), f(y)) &= d_1\big((\tfrac{1}{2}x_2, \tfrac{1}{2}(x_1 + 1)), (\tfrac{1}{2}y_2, \tfrac{1}{2}(y_1 + 1))\big) \\ &= \big|\tfrac{1}{2}(x_2 - y_2)\big| + \big|\tfrac{1}{2}(x_1 - y_1)\big| \\ &= \tfrac{1}{2}d_1(x, y). \end{aligned}$$

Hence f is a contraction, with $c = \frac{1}{2}$.
The fixed point is given by the solution of the equations $x_1 = \frac{1}{2}x_2$ and $x_2 = \frac{1}{2}(x_1 + 1)$, which is $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$. The fixed point is thus $x = (\frac{1}{3}, \frac{2}{3})$.

B3. (a) [Standard definition.] A map $f : X \to Y$ is continuous at $\alpha \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\sigma(f(x), f(\alpha)) < \varepsilon$ whenever $\varrho(x, \alpha) < \delta$. The map f is said to be *continuous* if it is continuous at every point $\alpha \in X$.

(b) [Bookwork.] Assume that f is continuous and that $x_n \to \alpha$ in (X, ϱ). We need to show that for every $\varepsilon > 0$ there exists N_ε such that

$$f(x_n) \in B_\varepsilon(f(\alpha)), \quad \text{for all } n > N_\varepsilon. \tag{1}$$

Since f is continuous, given $\varepsilon > 0$ we can find $\delta > 0$ such that $f(B_\delta(\alpha)) \subseteq B_\varepsilon(f(\alpha))$. For this δ, since the sequence x_n converges to α in (X, ϱ), there exists a number N_δ such that $x_n \in B_\delta(\alpha)$ for all $n > N_\delta$. Then $f(x_n) \in f(B_\delta(\alpha)) \subseteq B_\varepsilon(f(\alpha))$, for $n \ge N_\delta$; in other words, (**??**) holds true with $N_\varepsilon = N_\delta$.

(c) [Bookwork.] Let y_n be an arbitrary sequence of elements of $f(K)$. Then $y_n = f(x_n)$ where $x_n \in K$. Since K is compact, the sequence x_n has a subsequence x_{n_k} which converges to a limit $\alpha \in K$. Then by part (b) the subsequence $y_{n_k} = f(x_{n_k})$ converges to the limit $f(\alpha) \in f(K)$. This proves that $f(K)$ is compact.

[This question continues overleaf . . .]

(d) [Unseen.] $f^{-1}(L) = \{x \in X : f(x) \in L\}$. A compact subset is closed, and the inverse image of a closed set under a continuous function is closed. So $f^{-1}(L)$ is necessarily closed. However, it is not necessarily bounded: consider the function $f : \mathbb{R} \to \mathbb{R}$ that is the constant 0. ($\mathbb{R}$ has the usual metric.) Then $L = \{0\}$ is a compact set, but $f^{-1}(L) = \mathbb{R}$ which is not bounded. Because it is not bounded it is not compact either.

[End of examination paper]